Non-Commutative Abelian Higgs model

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We investigate the Non-Commutative Abelian Higgs model. We argue that it is possible to introduce a consistent renormalization method by imposing the Non-commutative BRST invariance of the theory and by introducing the Non-Commutative Quasi-Classical Action Principle.

Keywords: Non-Commutative gauge theories, Non-Commutative Abelian Higgs model, Seiberg-Witten map, Quantum Action Principle, Quasi-Classical Action Principle

INTRODUCTION

Recently some efforts have been done to investigate the renormalizability of the Non-Commutative (NC) field theories [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. In this paper we consider a possible field theory candidate for the non-commutative extension of the Abelian Higgs model. In particular we will show that the Non-Commutative Abelian Higgs (*NCAH*) model is stable under quantum corrections and then it is renormalizable in the sense of [11]. In ordinary commutative theories, the main ingredient used to prove the renormalizability is given by the Quantum Action Principle [12, 13, 14, 15]. As shown in [16, 17] the Quantum Action Principle (QAP), that affirms that the breaking of the Slavnov-Taylor is given by the insertion of a local operator, is a very strong requirement that can be relaxed into the Quasi-Classical Action Principle (QCAP) in the proof of the renormalizability of theories. The QCAP for general theories assumes that *the first non-vanishing order* in the loop expansion of the breaking of the Slavnov-Taylor is a local formal power series in the fields and external sources and their derivatives. Under the assumption that the QCAP [17] is valid for theories related to power-counting renormalizable theories we can inferred, by using the Seiber-Witten (SW) map, that the the QCAP is a valid proprieties of non-commutative theories too. We show that it is possible to formulate such theories, by using the Slavnov-Taylor identities (STi) related to the non-commutative extension of the gauge invariance. In particular we will analyse the *NCAH model*, and we will study the renormalization properties of it. By

using the NC extension of the QCAP and the SW map we find that the *NCAH model* is renormalizable in the sense that it needs a finite set of renormalization constant to become a well defined theory [11]. The *NCAH model* is as well predictable as the ordinary Abelian Higgs model. The plan of the letter is the following: First we show how to generalize the STi to NC theories; then we introduce the meaning of the QCAP and and we extent is to NC theories. After that we show how, by using the STi and the Non-Commutative Quasi-Classical Action Principle, it is possible to study the renormalizability of the *NCAH model*.

SLAVNOV-TAYLOR IDENTITIES FOR NON-COMMUTATIVE THEORIES

Let us now give a look the the STi and let us generalize them to NC theories. First, as usually, given a fields transformation s:

$$s\phi_i = R_{\alpha i}c_\alpha, \tag{1}$$

where R are generally formal power series of the fields and of their derivatives, and c_α are Grassmann fields which transforms, for example in the case of gauge symmetries, according to

$$sc_a = -\frac{1}{2}f_a{}^{bc}c_bc_c, \tag{2}$$

we couple the non linear transformation to additional external sources ρ and add the new terms to the classical action. Then the invariance can be written in a functional form as:

$$\mathcal{S}(\Gamma^{(0)}) = \int d^4x \left(\frac{\delta\Gamma^{(0)}}{\delta\rho_i}\frac{\delta\Gamma^{(0)}}{\delta\phi_i}\right) = 0. \tag{3}$$

The real meaning of the functional derivative is that anywhere appear the field Y one has to substitute it with X. This is an important subtlety when treating with non commutative theories. According to this full meaning we can safely change the ordinary product into the $\star$-product and forget about the previous meaning but use the operative standard rule: $X \star \dfrac{\delta}{\delta Y}$ is for *left* functional derivatives, while $\dfrac{\delta}{\delta Y} \star X$ is for *right* functional derivatives. For these reasons, the NC extension of the STi in eq. (3) is:

$$\mathcal{S}(\Gamma^{(0)\star}) = \int d^4x \left(\frac{\delta\Gamma^{(0)\star}}{\delta\rho_i} \star \frac{\delta\Gamma^{(0)\star}}{\delta\Phi_i}\right) = 0. \tag{4}$$

Operatively, we also remember the fundamental rules of an integrated $\star$-product [21, 22]:

$$\int d^4x \, a \star b \star c = (-1)^F \int d^4x \, c \star a \star b \,, \tag{5}$$

where F is the fermi-bose factor, which take into account the fact that the fields are (anti)-commuting. Once we have clarified the Slavnov-Taylor identities for a Non-Commutative theory, we need a further step: the nilpotency of the linearized Slavnov-Taylor operator. This is a extension of the nilpotency of the ordinary linearized Slavnov-Taylor operator and it can be checked mechanically [18], by using the definition in eq. (4) and the properties of the $\star$-product.

THE QUASI-CLASSICAL ACTION PRINCIPLE

It is customary to summarize the content of the QAP by characterizing the behavior of the renormalized quantum effective action Γ under infinitesimal variations of the fields and the parameters of the model. For self consistency we report the full standard formulation (as given for instance in [17, 19]) in appendix . The QAP tells us that in a power-counting renormalizable theory the ST-like identity in eq. (11) can be broken at quantum level only by the insertion of an integrated local composite operator of bounded dimension. This is an all-order statement holding true regardless the normalization conditions chosen. At the lowest non-vanishing order the insertion reduces to a local polynomial in the fields and external sources and their derivatives with bounded dimensions. This property is a consequence of the topological nature of the $\hbar$-expansion as a loop expansion. That is, if a local insertion in the vertex functional were zero up to the order $n-1$, at the n-th order it must reduce from a diagrammatic point of view to a set of points. By power-counting this set is finite and hence it corresponds to a local polynomial in the fields and the external sources and their derivatives. The extension of the QAP beyond the power-counting renormalizable case is yet an open issue in the theory of renormalization. In [17] the QCAP have been introduced as a consequence of the Stora conjecture [16]. In the power-counting renormalizable case bounds on the dimensions can be given truncating the formal power series predicted by the QCAP to a local polynomial. Thus the QCAP reduces in this case to the part of the QAP stating that the lowest non-vanishing order $\Delta^{(n)}(x)$ of the breaking term is a polynomial. This justifies the name of QCAP. For most practical purposes the QCAP (or, for power-

counting renormalizable theories, the part of the QAP relevant to $\Delta^{(n)}(x))$ is what is really needed in order to carry out the program of Algebraic Renormalization. In particular, this is enough to discuss the restoration of anomaly-free ST-like identities order by order in the loop expansion. This point have been illustrated in [17] on the example of the quantization of the Equivalence Theorem ST identities.

The Non-Commutative theories

The existence of the SW map [1, 20], which for example affirms that it is possible to construct a map that relates an ordinary theory to a Non-Commutative one by conserving the gauge symmetry properties, allows us to write an extension of the QCAP to NC theories:

Proposition 1 *Let $\Gamma^\star$ be the vertex functional corresponding to a Non-Commutative theory with a classical action given by $\Gamma^{(0)\star} = \int d^4x \, \mathcal{L}^\star(\Phi_a, \beta_i, \lambda)$ where Φ_a are the quantum fields, β_i the external sources coupled to field polynomials Q^i, and λ stands for the parameters of the model (masses, coupling constants, renormalization points). Here the * indicates that every fields multiplications are assumed to enter in the Lagrangian via the NC product. Notice that $\Gamma^{(0)\star}$ is not power-counting renormalizable however the commutative corresponding one, $\Gamma^{(0)}$, gives up to a power-counting renormalizable theory. Let us define*

$$\mathcal{S}(\Gamma^\star) = \int d^4x \left(\alpha_a \frac{\delta\Gamma^\star}{\delta\Phi_a(x)} + \alpha_{ab}\Phi_b(x) \star \frac{\delta\Gamma^\star}{\delta\Phi_a(x)} + \alpha_{ia}\frac{\delta\Gamma^\star}{\delta\beta_i(x)} \star \frac{\delta\Gamma^\star}{\delta\Phi_a(x)} + \alpha\frac{\delta\Gamma^\star}{\delta\lambda} \right). (6)$$

Then (**Non-Commutative Quasi Classical Action Principle**) *the first non-vanishing order in the loop expansion, say n, of $\mathcal{S}(\Gamma^\star)$:*

$$\int d^4x \, \Delta^{(n)\star} \equiv \mathcal{S}(\Gamma^\star)^{(n)}, \qquad \mathcal{S}(\Gamma^\star)^{(j)} = 0 \text{ for } j = 0, 1, \ldots, n-1 \qquad (7)$$

is an integral of a formal power series in the fields and external sources and their derivatives with given commutative bounded dimensions but multiplied via the star product.

Notice that in our specific case, being the Slavnov-Taylor operator nilpotent, the breaking term must satisfy the Wess-Zumino consistent condition: $\mathcal{S}(\int d^4x \, \Delta^{(n)\star}) = 0$.

THE NON-COMMUTATIVE ABELIAN HIGGS MODEL

As it is well know, the Abelian Higgs model comes out form a global $O(2)$ symmetric ϕ^4 theory coupled to a local $U(1)$ Abelian gauge theory within the minimal coupling. It is assumed that the potential for the complex ϕ field has the form of a Mexican hat and that the true perturbative minimum of the potential is obtained when the ϕ acquire a non trivial vacuum expectation value. Written in term of the real and imaginary part of the ϕ the Lagrangian does not show any more the hidden global symmetry, however the UV properties are not modified by this dynamical symmetry breaking. In particular the theory turns out to be power-counting renormalizable. The Non-Commutative equivalent of the Abelian Higgs model can be obtained by substituting everywhere in the Abelian Higgs model, in the unbroken phase, the ordinary product with the $\star$-product. Once the field ϕ acquires a vacuum expectation value, the action become more involved, however it can be easily worked out [10]. By solving the Wess-Zumino consistent condition, it is possible to show that the Non-Commutative Abelian Higgs model is renormalizable and that we need to introduce the renormalization of the fields A, Φ, c, and of the constants μ, λ, g and of the tadpole ν. This is in accordance with the one loop computation done in [10].

THE QUANTUM ACTION PRINCIPLE

The standard formulation of the QAP is the following:

Proposition 2 *Let Γ be the vertex functional corresponding to a (power-counting renormalizable) theory in a D-dimensional space-time with a classical action given by*

$$\Gamma^{(0)} = \int d^D x \, \mathcal{L}(\varphi_a, \beta_i, \lambda) \tag{8}$$

where φ_a are the quantum fields, β_i the external sources coupled to field polynomials Q^i and λ stands for the parameters of the model (masses, coupling constants, renormalization points). Let the inverse of the quadratic part of the action be the standard Feynman propagators. Given the local operator

$$\mathcal{S}(\Gamma) \equiv \alpha_a \frac{\delta\Gamma}{\delta\varphi_a(x)} + \alpha_{ab}\varphi_b(x)\frac{\delta\Gamma}{\delta\varphi_a(x)} + \alpha_{ia}\frac{\delta\Gamma}{\delta\beta_i(x)}\frac{\delta\Gamma}{\delta\varphi_a(x)} + \alpha\frac{\delta\Gamma}{\delta\lambda}, \tag{9}$$

where $\alpha_a, \alpha_{ab}, \alpha_{ia}$ and α are constants, then the Quantum Action Principle can be stated in the following way

$$\mathcal{S}(\Gamma) = \Delta(x) \cdot \Gamma = \Delta^{(n)}(x) + O(\hbar^{n+1}). \tag{10}$$

$\Delta(x) \cdot \Gamma$ denotes the insertion of a local operator. Moreover the lowest non-vanishing order coefficient $\Delta^{(n)}(x)$ of $\Delta(x) \cdot \Gamma$ is a local polynomial in the fields and external sources and their derivatives with bounded dimension.

At the integrated level (Slavnov-Taylor-like identities) the QAP reads

$$\mathcal{S}(\Gamma) \equiv \int d^4x \, \mathcal{S}(\Gamma) = \int d^4x \, \Delta(x) \cdot \Gamma \,. \tag{11}$$

The first non-vanishing order of the ST-like breaking terms is given by

$$\Delta^{(n)} \equiv \mathcal{S}(\Gamma)^{(n)} = \int d^4x \, \Delta^{(n)}(x) \,. \tag{12}$$

As a consequence of Proposition 2, $\Delta^{(n)}$ is an integrated local polynomial in the fields and the external sources and their derivatives with bounded dimension. The ultraviolet (UV) dimension d_Δ of $\Delta^{(n)}$ can be predicted from the UV dimensions d_a of the fields φ_a and the UV dimensions d_{Q^i} of the field polynomials Q^i [19]. We do not dwell on this problem here since the only information we need for the present discussion is the fact that d_Δ is bounded.

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[1] M. Picariello, A. Quadri and S. P. Sorella, JHEP **0201**, 045 (2002)

[2] G. Barnich, F. Brandt and M. Grigoriev, Nucl. Phys. B **677**, 503 (2004)

[3] R. Amorim and F. A. Farias, Phys. Rev. D **69**, 045013 (2004)

[4] F. Brandt, C. P. Martin and F. R. Ruiz, JHEP **0307**, 068 (2003)

[5] C. P. Martin, Nucl. Phys. B **652**, 72 (2003)

[6] F. Ruiz Ruiz, Nucl. Phys. B **637**, 143 (2002)

[7] I. Y. Aref'eva, D. M. Belov and A. S. Koshelev, Phys. Lett. B **476**, 431 (2000)

[8] I. Chepelev and R. Roiban, "Convergence theorem for non-commutative Feynman graphs and JHEP **0103**, 001 (2001)

[9] L. Griguolo and M. Pietroni, Phys. Rev. Lett. **88**, 071601 (2002)

[10] F. J. Petriello, Nucl. Phys. B **601** (2001) 169

[11] J. Gomis and S. Weinberg, Nucl. Phys. B **469**, 473 (1996)

[12] Y.-M. P. Lam, *Phys. Rev.* **D7** (1973) 2943–2949.

[13] M. C. Bergere and Y.-M. P. Lam, *Phys. Rev.* **D13** (1976) 3247.

[14] P. Breitenlohner and D. Maison, *Commun. Math. Phys.* **52** (1977) 11.

[15] T. E. Clark and J. H. Lowenstein, *Nucl. Phys.* **B113** (1976) 109.

[16] R. Stora, *Talk delivered at the Conference on Mathematical Physics*, Certosa di Pontignano - Siena June 20-25 2000.

[17] R. Ferrari, M. Picariello and A. Quadri, JHEP **0204**, 033 (2002)

[18] M. Picariello and E. Torrente-Lujan, Comput. Phys. Commun. **156**, 171 (2004)

[19] O. Piguet and S. P. Sorella, Berlin, Germany: Springer (1995) 134 p.

[20] N. Seiberg and E. Witten, JHEP **9909** (1999) 032

[21] R. J. Szabo, Phys. Rept. **378** (2003) 207

[22] M. R. Douglas and N. A. Nekrasov, Rev. Mod. Phys. **73**, 977 (2001)